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LONDON

TECHNICAL REPORT
ONRL-35-54

COLLOQUIUM ON LINEAR EQUATIONS

American Embassy

London, England

OFFICE OF NAVAL RESEARCH
LONDON

TECHNICAL REPORT
ONRL-35-54

COLLOQUIUM ON LINEAR EQUATIONS

Summary

This report presents a summary of some of the work presented at a Colloquium on Linear Equations and Inequalities, held at Oberwolfach, Germany in October 1953. This work includes: an application of the Stiefel-Rutishauser quotient-difference algorithm which involves a novel relation of an inverse power series with a continued fraction; a method presented by Prof. Stiefel applying orthogonal polynomial sets to the approximate solution of symmetric, positive-definite systems; an elementary proof by Dr. Ehlers of the inequality $\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} a_i$, where the a_i's are positive real; a description by Dr. Unger of the Jenne-Friedrich method of evaluating determinants; and by the writer, a variant of the classical Schlömilch expansion of an arbitrary function in a series of J_0 functions.

21 June 1954

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COLLOQUIUM ON LINEAR EQUATIONS

INTRODUCTION

During the week 5 to 10 October 1953, a Colloquium on Linear Equations and Inequalities was held at the Mathematical Research Institute at Oberwolfach, in the Black Forest. This Colloquium had the same informal and pleasant nature characteristic of such meetings at Oberwolfach, and was attended by about twenty mathematicians, mostly from German and Swiss universities. A list of the participants is given in Appendix I.

Since a thorough description of the Institute, together with a detailed review of its history, has been given in Technical Report ONRL-12-52, this will not be repeated. Briefly, the Institute is housed in a building known as the Lorenzenhof, located at Oberwolfach-Walke, Schwarzwald, about forty miles from Freiburg i/Br. The history of Oberwolfach starts with the organization by Professor W. Süss in the middle of 1944 of a Central Mathematics Institute for the purpose of acting as a national research and consultation center. The organization was divided into two parts, pure mathematics and applied mathematics, and was housed in the Lorenzenhof. The policy of the Institute as established very early in its existence was to employ as many good people as possible for research along lines which would produce mathematics and mathematicians of lasting value irrespective of the applications of the moment. During the allied occupation, the Institute served as a refuge for mathematicians in distress. In the surroundings at Oberwolfach, their interest in mathematics had an opportunity to be reawakened.

Ever since the fall of Germany, the finances of the Institute have been rather precarious. The receiving of the assignment from the military government to prepare the FIAT reviews on pure mathematics helped, principally in the matter of food, and the Institute has for the last several years received a small annual allocation from the Land Baden, enough to cover rent and a few basic expenses. The situation of Oberwolfach is at present rather clouded and it is not at all certain that it will continue to get even the nominal support essential for its bare existence. The reason for this lies in a recent political change in Germany, the consolidation of two of the Länder into one. The former Länder of Baden and Württemberg have been combined into a single one, with, of course, corresponding changes in the Departments of Education. The new Ministry of Education has now to be convinced anew of the value of the Institute to the Land and to the Federal Republic.

During most of the week of the Colloquium three meetings a day were held, one in the morning, one in the late afternoon and one in the early evening. Some of the twelve or fifteen contributions presented at Oberwolfach are discussed below. In keeping with the informal and "workshop" nature of the meeting, much of the work presented was tentative or unpolished in character and much of the time was spent in critical discussion. No time limit was ever imposed.

AN APPLICATION OF THE QUOTIENT-DIFFERENCE ALGORITHM

A quotient-difference procedure for the investigation of eigenvalue problems was presented at the 1953 meeting of GAMM in Aachen by Professor E. Stiefel of the E.T.H. in Zürich. This procedure was reported in detail in Technical Report ONRL-76-53. At Oberwolfach, Dr. H. Rutishauser, who collaborated with Prof. Stiefel in the development of the algorithm, presented an application of this algorithm to the problem of obtaining solutions to linear equation systems. This application involves the use of continued fractions.

The quotient-difference scheme starts with an original set of numbers designated by f_i^1 and follows the equations

$$Q_i^k = f_{i+1}^k / f_i^k$$

$$\Delta_i^k = Q_{i+1}^k - Q_i^k$$

$$\bar{\Delta}_i^k = \Delta_i^k + \bar{\Delta}_{i+1}^{k-1}$$

$$f_i^{k+1} = f_{i+1}^k \bar{\Delta}_i^k$$

If f_i^1 has the form $\sum_{j=1}^{r} a_j \lambda_j^i$, then $\bar{\Delta}_i^r \equiv 0$. If the λ_j are positive real, distinct, and arranged in order of descending magnitude, then $Q_i^k \to \lambda_k$ as $i \to \infty$.

The method presented by Dr. Rutishauser is based on a relationship afforded by the quotient-difference algorithm between a certain inverse power series and a continued fraction. This relation is given

$$f(\lambda) = \sum_{i=0}^{\infty} \frac{f_i^1}{\lambda^{i+1}} = \frac{f_0^1}{\lambda} + \frac{f_1^1}{\lambda^2} + \cdots$$

$$= \cfrac{f_0^1}{\lambda - 1 - \cfrac{\bar{\Delta}_0^1}{\lambda - 1 - \cfrac{Q_0^2}{\lambda - 1 - \cfrac{\bar{\Delta}_0^2}{\lambda - 1 - \cfrac{Q_0^3}{\lambda - 1 - \cdots}}}}}$$

This method may in some cases be useful for the numerical evaluation of semi-convergent series.

The particular application is to the problem of solving the equation system

$$Ax = r$$

expressed here in matrix form. A formal solution is

$$x = [\lambda E - (\lambda E - A)]^{-1} r = \frac{r}{\lambda} + \frac{(\lambda E - A) r}{\lambda^2} + \cdots\cdots$$

convergent if A has only positive real eigenvalues and if λ is sufficiently large. If f_i' represents a particular element of r_i, where

$$r_0 = r$$

$$r_{i+1} = (\lambda E - A) r_i$$

then the corresponding element of x is given by

$$f(\lambda) = \sum_{i=0} \frac{f_i'}{\lambda^{i+1}} = \frac{f_0'}{\lambda} \; \underset{1}{\overset{Q_0'}{-}} \; \underset{\lambda}{\overset{\Delta_0'}{-}} \; - \cdots$$

The procedure is to pick a convenient value of λ, calculate the r_i, carry out the quotient difference procedure on a particular element, and to calculate the corresponding element of x through the continued fraction expression. This continued fraction may be convergent even though the equivalent inverse power series is not.

As an example, the negative of the Laplace operator is considered in the usual finite difference form, applied to a square region with homogeneous boundary conditions

```
0-0-0-0-0-0
0 x x x x 0
0 x x x x 0
0 x x x x 0
0 x x x x 0
0-0-0-0-0-0
```

$$A = \begin{matrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{matrix}$$

The quantity λ is chosen to be equal to 5, and the operator $\lambda E - A$ is expressed

$$\lambda E - A = \begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix}$$

The distribution r for which a solution is desired is chosen

$$r = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} = r_0$$

for which

$$r_1 = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} \qquad r_2 = \begin{matrix} 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 5 & 2 & 1 \\ 2 & 2 & 2 & 0 \end{matrix} \qquad r_3 = \begin{matrix} 3 & 3 & 3 & 0 \\ 6 & 12 & 6 & 3 \\ 11 & 13 & 12 & 3 \\ 6 & 11 & 6 & 3 \end{matrix}$$

$$r_4 = \begin{matrix} 12 & 21 & 12 & 6 \\ 32 & 40 & 36 & 12 \\ 36 & 59 & 40 & 21 \\ 28 & 36 & 32 & 12 \end{matrix} \qquad r_5 = \begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 133 & \cdot & \cdot \\ 155 & 211 & 133 & \cdot \\ \cdot & 155 & \cdot & \cdot \end{matrix}$$

The quotient-difference procedure is applied to the element for which r was non-zero.

1							
	1						
1		5	5				
	5			$-\frac{12}{5}$			
5		$-\frac{12}{5}$	-12		$\frac{3}{10}$	$-\frac{21}{10}$	$\frac{126}{5}$
	$13/5$			$-\frac{21}{10}$			$\frac{25}{2\cdot7}$
13		$\frac{126}{5\cdot13}$	$\frac{126}{5}$		$\frac{-139}{2\cdot5\cdot7\cdot13}$	$\frac{25}{14}$	$\frac{315}{7}$
	$59/13$			$-\frac{5\cdot41}{13\cdot7}$			
59		$-\frac{738}{13\cdot59}$	$-\frac{738}{13}$				
	$211/59$						

and x for that element is expressed

$$x = \frac{1}{5} - \frac{1}{1} - \frac{5}{5} - \frac{-\frac{12}{5}}{1} - \frac{-\frac{21}{10}}{5} - \frac{\frac{25}{14}}{1} - \cdots$$

The successive approximations to the particular element of interest are

$$x = \frac{1}{5}, \frac{1}{4}, 0, \frac{12}{23}, \frac{24}{25}, \frac{9}{5}, \cdots$$

It may be seen that, with the number of terms taken, no reasonable approximation has been reached.

An alternate method that might decrease the amount of calculation required in the quotient-difference algorithm would be to separate the first m terms, giving

$$f(\lambda) = \frac{f_0'}{\lambda} + \frac{f_1'}{\lambda^2} + \frac{f_2'}{\lambda^3} + \cdots + \frac{f_{m-1}'}{\lambda^m} + \frac{1}{\lambda^m}\left[\frac{f_m'}{\lambda - \frac{Q_m'}{1 - \frac{\Delta_m'}{\lambda - \frac{Q_m'}{1 - \cdots}}}}\right]$$

This type of variation to the method is necessary in case one or more of the elements in the partial fraction expression are zero (or infinite).

SOLUTION OF SYMMETRIC, POSITIVE-DEFINITE SYSTEMS OF EQUATIONS USING ORTHOGONAL POLYNOMIALS

Professor E. Stiefel of the E.T.H. in Zürich presented a method for obtaining approximate solutions of finite systems of equations in which the matrix is symmetric and positive definite. The basic equation system is expressed

$$Ax = k \tag{1}$$

and the eigenvalues of the matrix are presumed to be less than unity and arranged in order of increasing magnitude

$$0 < \lambda_1 \leq \lambda_2 < \lambda_3 \cdots < \lambda_n < 1$$

Since an upper bound to the eigenvalues of a symmetric matrix may be found, this requirement can always be satisfied by an appropriate reduction. For convenience, the case of distinct eigenvalues is considered.

An approximation scheme is envisioned, in which $x^{(m)}$ represents the m^{th} approximation to the solution, and the m^{th} residual vector is defined

$$r^{(m)} = k - Ax^{(m)} \tag{2}$$

The general equation for improving the solution is taken to be of the form

$$x^{(m+1)} - x^{(m)} = c_m (x^{(m)} - x^{(m-1)}) + a_m r^{(m)} \tag{3}$$

from which the equation for $r^{(m)}$ is obtained

$$r^{(m+1)} = (1 + c_m - a_m A)\, r^{(m)} - c_m r^{(m-1)} \tag{4}$$

The process is begun by setting

$$x^{(0)} = 0 \qquad c_0 = 0$$

The vector $r^{(m)}$ may be expressed in the form

$$r^{(m)} = R_m(A)\, k \tag{5}$$

where R_m is a polynomial of degree m in A. It may be noted that $R_m(0) = 1$. If the vector k is expressed

$$k = k_1 e_1 + k_2 e_2 + \cdots + k_n e_n \tag{6}$$

in terms of the normalized eigenvectors e_i, the vector $r^{(m)}$ is expressed

$$r^{(m)} = R_m(\lambda_1) k_1 e_1 + \cdots + R_m(\lambda_n) k_n e_n \tag{7}$$

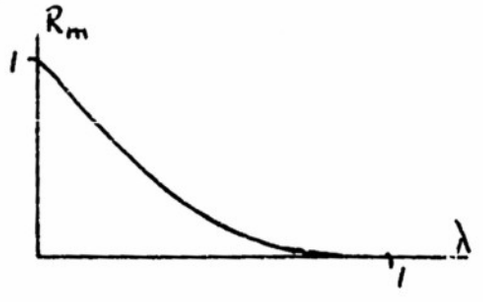
As an example the simple iteration scheme corresponding to the expansion

$$A^{-1} = \sum_{m=0} (E - A)^m$$

is considered. This fits into the general scheme with $c_m = 0$, $a_m = 1$. The polynomial R_m is given simply

$$R_m(\lambda) = (1-\lambda)^m$$

It is clear by considering a plot of $R_m(\lambda)$ that this gives rather poor approximations for a given order of iteration unless all the eigenvalues are close to unity.



It is clear that an approach using one of the well-known sets of polynomials related by recurrence relations and giving good representations of other functions within a fixed interval may be used to devise an interation scheme. As an example, with a lower bound λ_0 to the eigenvalues, the Tchebycheff polynomials may be used, with

$$R_m(\lambda) = \frac{T_m\left(\frac{1+\lambda_0-2\lambda}{1-\lambda_0}\right)}{T_m\left(\frac{1+\lambda_0}{1-\lambda_0}\right)}$$

These functions are orthogonal over the interval λ_0 to 1 with the weighting function

$$\rho(\lambda) = \frac{1}{\sqrt{(1-\lambda)(\lambda-\lambda_0)}}$$

If the eigenvalues were known, a procedure which led to a residual function $R^{(n)}$ which had zeros at the eigenvalues would yield an exact solution. This would be best done for a given k by constructing a set of orthogonal polynomials for $R^{(m)}$ with the weighting function

$$\rho(\lambda) = k_1^2 \delta(\lambda-\lambda_1) + \cdots \qquad + k_n^2 \delta(\lambda-\lambda_n) \tag{8}$$

where δ denotes the Dirac delta function, by means of which $R^{(n)}(\lambda)$ would automatically possess the desired properties. But the orthogonal property

$$\int_0^1 R_i(\lambda) R_j(\lambda) \rho(\lambda) d\lambda = 0 = (r^{(i)}, r^{(j)}) \quad \text{for } i \neq j \tag{9}$$

is equivalent in this case to the corresponding orthogonal properties for the residual vectors themselves. Thus the equations (4), together with the orthogonality conditions

$$\begin{aligned} (r^{(m+1)}, r^{(m)}) &= 0 \\ (r^{(m+1)}, r^{(m-1)}) &= 0 \end{aligned} \tag{10}$$

from which the values of am and cm may be determined, are able to accomplish this construction with no necessity for specific information about the actual eigenvalues. This is the principal result of Stiefel's contribution. The double orthogonality condition, together with Eq. (4) ensures orthogonality in general. The actual solution is obtained through Eq. (3).

It may be desired to use approximate information about the eigenvalue distribution and about the vector k in place of the foregoing specialized procedure. If the density of the eigenvalue distribution is given by $\mu(\lambda)$ and the component distribution of the vector k is given by $k(\lambda)$, the proper choice of density function in the orthogonal polynomial procedure is

$$\rho(\lambda) = \mu k^2 \tag{11}$$

With suitable choices for $\rho(\lambda)$, several of the well-known systems of orthogonal polynomials are obtained.

COMPARISON OF EQUATION-SOLVING METHODS

Dr. H. Unger of the Institut für Praktische Mathematik in the T.H. at Darmstadt discussed a number of standard methods for solving the equation system

$$Ax = a$$

in matrix notation. The general conclusion presented was that the Gauss algorithm, by which A is factored into two triangular matrices

$$A = QG \qquad \begin{pmatrix} Q \text{ of form } \blacktriangleleft \\ G \text{ of form } \blacktriangleright \end{pmatrix}$$

requires the least number of numerical operations.

Unger continued with a discussion of some of the variants of the Gauss algorithm.

(a) If the solution obtained is approximate, an improvement may be obtained by obtaining the residual

$$d_0 = Ax_0 - a$$

solving the incremental equation

$$A\,\Delta x_0 = -d_0$$

using the approximate triangular matrices, correcting x, and repeating as necessary.

(b) The splitting of A may be accomplished using diagonal matrices between the triangular matrices, in order to decrease the number of divisions which need to be carried out in a numerical inversion.

(c) The Gauss algorithm may be carried out with submatrices in place of the usual elements; this requires the inverses of the submatrices lying along the diagonal, themselves obtainable by the Gauss algorithm.

(d) If a number of solutions involving the same matrix are to be obtained, it is easier to carry out the calculation in terms of the inverses of the triangular matrices

$$x_i = G^{-1}Q^{-1}a_i$$

than to calculate A^{-1} and obtain x_i directly.

ELEMENTARY PROOF OF AN INEQUALITY

Dr. G. Ehlers of the University of Kiel gave a simple proof of the inequality

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leqq \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

where the a's are all positive real. Noting that the expression is homogeneous, it may be required that

$$a_1 a_2 \cdots a_n = 1$$

and it is then necessary to prove that

$$a_1 + a_2 + \cdots + a_n \geqq n$$

The proof follows the method of induction.

Assuming the inequality holds for n and that

$$a_1 a_2 \cdots a_n a_{n+1} = 1$$

then

$$a_1 a_2 + a_3 + \cdots + a_n + a_{n+1} \geqq n$$

where a_1 and a_2 are any two out of the set of a's. Choosing one of these greater than one and the other less than one ($a_i \equiv 1$ is a trivial case),

$$(a_2 - 1)(a_1 - 1) \leqq 0$$

combining the two inequalities yields

$$a_1 + a_2 + \cdots + a_n + a_{n+1} \geqq n + 1$$

SOLUTION OF LINEAR EQUATIONS BY COMPLEX CONTOUR INTEGRALS

Dr. G. Ehlers presented the solution of the equation system

$$\sum_{i=1}^{r} A_i X B_i = Y$$

where the elements involved are square matrices and the matrices B_i are commutable. The solution is

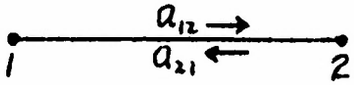
$$X = \frac{1}{(2\pi i)^r} \oint_{B_1} \oint_{B_2} \cdots \oint_{B_r} \frac{1}{\sum A_i z_i} Y \frac{dz_1}{z_1 - B_1} \cdot \frac{dz_2}{z_2 - B_2} \cdots \frac{dz_r}{z_r - B_r}$$

where $\oint_{B_i}$ goes around the entire spectrum of B_i but not around any singularity of $\sum A_i z_i$.

THE JENNE-FRIEDRICH METHOD OF EVALUATING DETERMINANTS

Dr. H. Unger of Darmstadt described a method for the evaluation of determinants which, existent in less available literature, is relatively unknown. The original reference is a paper by K. Friedrich, Stuttgart, 1930, or in a Sonderabdruck aus Zeit. f. Vermess.-Wesen, 1930. A second reference is by W. Jenne, "Zur Auflösung linearer Gleichungs-systeme", Astronomische Nachrichten 278, 73 (1949). The method is based upon a lattice-like graphical representation of the terms of a determinant.

Each diagonal term of the matrix or determinant is represented by a point, for example the term a_{22} by the point 2. Each conjugate pair of off-diagonal terms is represented by a line joining the corresponding points, with each term associated with a direction along this line; for example,



Additional notation is needed. Parentheses are used to denote a complete diagonal subdeterminant (one whose diagonal terms are diagonal terms of the original determinant) or a single diagonal term. The expression

$$|\mu\nu\pi \cdots \varkappa\lambda\mu| = a_{\mu\nu} a_{\nu\pi} \cdots\cdots a_{\varkappa\lambda} a_{\lambda\mu}$$

is defined also. Note that $|\mu| = a_{\mu\mu} = (\mu)$.

The determinant is considered to be evaluated with respect to a certain point, μ , and is given by the expression

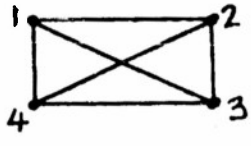
$$D = \sum_{r=1}^{\text{all perm.}} |\mu\nu \cdots \lambda\mu| \; R_{\mu,\nu,\cdots\lambda,\mu} \; (-1)^r$$

where r is the number of terms in the $|\ |$ expression, and R is the diagonal sub-determinant with terms in $\mu,\nu,\cdots\lambda$ missing. The summation is taken over all permutations of μ with any number of other indices (including none).

Some clarification may be brought about by an example. The complete 4×4 determinant

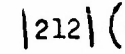
$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

is denoted by its lattice representation

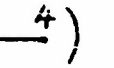
$D =$ 

Evaluating with respect to the point 2,

$D =$ (2) 

$- |212|$ 

$- |232|$ 

$- |242|$ 

$+ |2132|\ (4) \quad + |2312|\ (4) \quad + |2142|\ (3)$

$+ |2412|\ (3) \quad + |2342|\ (1) \quad + |2432|\ (1)$

$- |21342| \quad - |21432| \quad - |23142| \quad - |23412|$

$- |24132| \quad - |24312|$.

It is evident that this procedure presents no attraction over conventional methods in the general case. Its only claim to usefulness is for cases in which certain of the off-diagonal pairs are zero, in which certain of the connecting lines of the lattice are broken. In the example above, if $a_{31} = a_{31} = a_{41} = a_{41} = 0$, the results quoted above take the form

$D =$ 

$= (2)(1)($ 

$)$

$-|212|($

$) - |232|(1)(4) - |242|(1)(3)$

$+|12342|(1) + |12432|(1).$

If the expansion is made relative to the point 1, the expression is simply

$D = (1)($ 

$) - (121)($

$)$

In the particular case of the determinant of a Jacobi matrix the lattice has the simple form

In a comparison of the Jenne-Friedrich Lattice Method applied to the treatment of matrix and determinant problems with the Gauss algorithm, Dr. Unger pointed out that even in the case of a Jacobi matrix the Gauss procedure is better. Only in the case for which the sub- and super-diagonal terms are all 1 or -1 does the Lattice Method show any numerical advantage. Interest in the method must probably depend on the conceptual picture it gives of a determinant and the fact that it furnishes a numerical expansion symmetric with respect to the diagonal.

NEO-SCHLÖMILCH SERIES

Dr. W. Hayes of ONR London presented a variant of the classical Schlömilch expansion of an arbitrary function as a series of Bessel functions. In place of the classical variation of the argument proportional to an integer m the variation is as $\sqrt{k^2+m^2}$ where k is constant. Series of this type appear in the theory of solutions of the equation $\phi_{xx} + \phi_{yy} - \phi_{zz} - k^2\phi = 0$ which are periodic (of period 2π, for example) in the variable x.

The function to be expanded is

$$f(x) = \tfrac{1}{2}a_0 J_0(kx) + \sum_{m=1}^{\infty} a_m J_0\left(\sqrt{k^2+m^2}\,x\right) \qquad (0 \le x \le \pi)$$

the principal problem being to evaluate the coefficients a_m in terms of the function $f(x)$. If the function

$$g(x) = \tfrac{1}{2}a_0 + \sum a_m \cos mx$$

is known, the coefficients are then given directly by

$$a_m = \frac{2}{\pi}\int_0^{\pi} g(x)\cos mx\, dx$$

A modification of Parseval's integral shows that the functions f and g are related by the equation

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} g(x\sin\theta)\cos(kx\cos\theta)\, d\theta$$

(cf. Watson, G. N., "Theory of Bessel Functions", Cambridge, 1944, Sects. 2.2, 19.1, and 19.41), and the solution of this integral equation is

$$g(x) = f(0)\cosh kx + x\int_0^{\pi/2} f'(x\sin\varphi)\cosh(kx\cos\varphi)\, d\varphi$$

This determination of g in terms of f completes the problem. Requirements for convergence are the same as for classical Schlömilch series.

It may be noted that k may be imaginary, whereby in essence the cos and cosh functions in the above analysis are exchanged. In the Bessel function series a finite number of terms are then expressible with the modified Bessel function I_0 in place of J_0.

As with the classical Schlömilch series, a null series may be obtained. This series is

$$\tfrac{1}{2}J_0(kx) + \sum_{m=1}^{\infty}(-1)^m J_0\left(\sqrt{k^2+m^2}\,x\right) = 0 \qquad 0<x<\pi$$

It may be again noted that k need not be real in this series.

Appendix I

Leaders of the Colloquium:

Ostrowski, Prof. A., Basel
Süss, Prof. W., Freiburg

Participants:

Bauer, Dr. F., München
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Ceschino, Dr. F., Paris
Collatz, Prof. L., Hamburg
Ehlers, Dr. G., Kiel
Gericke, Prof. H., Freiburg
Görtler, Prof. H., Freiburg
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Hayes, Dr. W., ONR London
von Hoerner, Dr. S., Göttingen
Hornecker, Dr. G., Paris
Kuntzmann, Prof. F., Grenoble
Ostmann, Prof. H. H., Berlin
Pisula, Dr. K., Göttingen
Rutishauser, Dr. H., Zürich
Stiefel, Prof. E., Zürich
Stucken, Dr. H., Neukirchen
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